OCR
RECOGNISING ACHIEVEMENT

Friday 13 January 2012 – Morning

A2 GCE MATHEMATICS

4726 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4726
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 Given that $f(x) = \ln(\cos 3x)$, find $f'(0)$ and $f''(0)$. Hence show that the first term in the Maclaurin series for $f(x)$ is ax^2, where the value of a is to be found. **[4]**

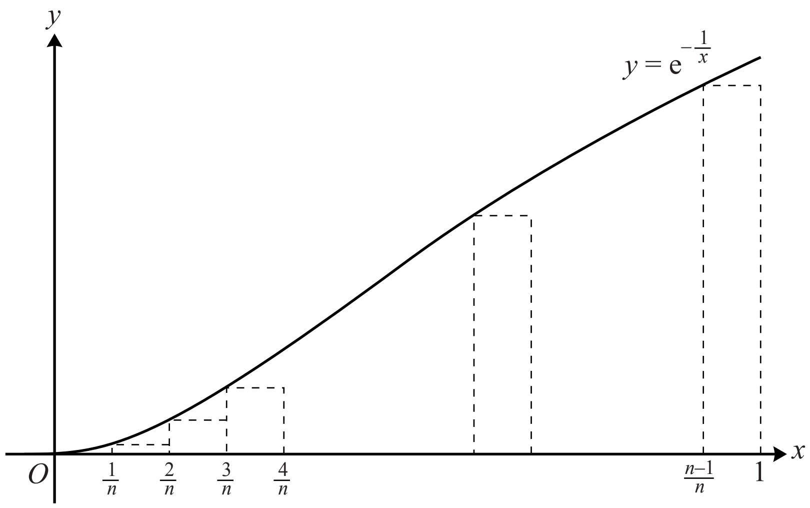
2 By first completing the square in the denominator, find the exact value of

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 - 4x + 5}\,dx\,.$$

[5]

3 Express $\dfrac{2x^3 + x + 12}{(2x-1)(x^2+4)}$ in partial fractions. **[7]**

4



The diagram shows the curve $y = e^{-\frac{1}{x}}$ for $0 < x \leqslant 1$. A set of $(n-1)$ rectangles is drawn under the curve as shown.

(i) Explain why a lower bound for $\int_0^1 e^{-\frac{1}{x}}\,dx$ can be expressed as

$$\frac{1}{n}\left(e^{-n} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + \ldots + e^{-\frac{n}{n-1}}\right).$$

[2]

(ii) Using a set of n rectangles, write down a similar expression for an upper bound for $\int_0^1 e^{-\frac{1}{x}}\,dx$. **[2]**

(iii) Evaluate these bounds in the case $n = 4$, giving your answers correct to 3 significant figures. **[2]**

(iv) When $n \geqslant N$, the difference between the upper and lower bounds is less than 0.001. By expressing this difference in terms of n, find the least possible value of N. **[3]**

5 It is given that $f(x) = x^3 - k$, where $k > 0$, and that α is the real root of the equation $f(x) = 0$. Successive approximations to α, using the Newton-Raphson method, are denoted by $x_1, x_2, \ldots, x_n, \ldots$.

(i) Show that $x_{n+1} = \dfrac{2x_n^3 + k}{3x_n^2}$. **[2]**

(ii) Sketch the graph of $y = f(x)$, giving the coordinates of the intercepts with the axes. Show on your sketch how it is possible for $|\alpha - x_2|$ to be greater than $|\alpha - x_1|$. **[3]**

It is now given that $k = 100$ and $x_1 = 5$.

(iii) Write down the exact value of α and find x_2 and x_3 correct to 5 decimal places. **[3]**

(iv) The error e_n is defined by $e_n = \alpha - x_n$. By finding e_1, e_2 and e_3, verify that $e_3 \approx \dfrac{e_2^3}{e_1^2}$. **[3]**

6 **(i)** Prove that the derivative of $\cos^{-1} x$ is $-\dfrac{1}{\sqrt{1-x^2}}$. **[3]**

A curve has equation $y = \cos^{-1}(1 - x^2)$, for $0 < x < \sqrt{2}$.

(ii) Find and simplify $\dfrac{dy}{dx}$, and hence show that

$$\left(2 - x^2\right)\frac{d^2y}{dx^2} = x\frac{dy}{dx} .$$

[5]

7 **(i)** Given that $y = \sinh^{-1} x$, prove that $y = \ln\left(x + \sqrt{x^2+1}\right)$. **[3]**

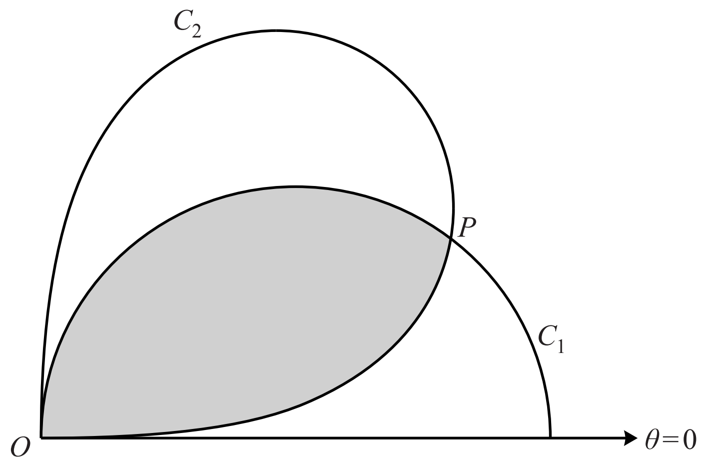
(ii) It is given that x satisfies the equation $\sinh^{-1} x - \cosh^{-1} x = \ln 2$. Use the logarithmic forms for $\sinh^{-1} x$ and $\cosh^{-1} x$ to show that

$$\sqrt{x^2+1} - 2\sqrt{x^2-1} = x .$$

Hence, by squaring this equation, find the exact value of x. **[5]**

[Questions 8 and 9 are printed overleaf.]

8



The diagram shows two curves, C_1 and C_2, which intersect at the pole O and at the point P. The polar equation of C_1 is $r = \sqrt{2}\cos\theta$ and the polar equation of C_2 is $r = \sqrt{2\sin 2\theta}$. For both curves, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The value of θ at P is α.

(i) Show that $\tan\alpha = \frac{1}{2}$. **[2]**

(ii) Show that the area of the region common to C_1 and C_2, shaded in the diagram, is $\frac{1}{4}\pi - \frac{1}{2}\alpha$. **[7]**

9 **(i)** Show that $\tanh(\ln n) = \dfrac{n^2 - 1}{n^2 + 1}$. **[2]**

It is given that, for non-negative integers n, $I_n = \displaystyle\int_0^{\ln 2} \tanh^n u \, du$.

(ii) Show that $I_n - I_{n-2} = -\dfrac{1}{n-1}\left(\dfrac{3}{5}\right)^{n-1}$, for $n \geqslant 2$. **[3]**

(iii) Find the value of I_3, giving your answer in the form $a + \ln b$, where a and b are constants. **[4]**

(iv) Use the method of differences on the result of part **(ii)** to find the sum of the infinite series

$$\frac{1}{2}\left(\frac{3}{5}\right)^2 + \frac{1}{4}\left(\frac{3}{5}\right)^4 + \frac{1}{6}\left(\frac{3}{5}\right)^6 + \dots .$$

[2]

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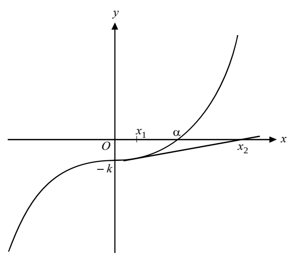
Question			Answer	Marks	Guidance	
1			$f'(x)=\frac{-3\sin 3x}{\cos 3x}=-3\tan 3x \Rightarrow f'(0)=0$	M1	For differentiating f(x) twice (y' as a function of a function)	
			$f''(x)=-9\sec^2 3x \Rightarrow f''(0)=-9$	A1	For correct f '(0) and f "(0) **www** (soi by correct expansion)	
			$\Rightarrow f(x)=-\frac{9}{2}x^2$	M1	For use of Maclaurin soi	If f"(0) = f'(0) = f(0) = 0 then **M0**
				A1	For correct series (condone $a=-\frac{9}{2}x^2$)	
			ALT: $\ln(\cos 3x)=\ln\left(1-\frac{1}{2}(3x)^2\right)=-\frac{9}{2}x^2$	**[4]**	**SC** Use of standard cos and ln series can earn second M1 A1	
				[4]		
2			$=\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{(2x-1)^2+4}\,dx$ *OR* $\frac{1}{4}\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{\left(x-\frac{1}{2}\right)^2+1}\,dx$	B1	For correct denominator (in 2nd case must include $\frac{1}{4}$)	
				M1	For integration to $k\tan^{-1}(ax+b)$ or $k\ln\left(\frac{ax+b-c}{ax+b+c}\right)$	
			$=\frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{2x-1}{2}\right]_{\frac{1}{2}}^{\frac{3}{2}}$ *OR* $\frac{1}{4}\left[\tan^{-1}\left(x-\frac{1}{2}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$	A1	FT for $ax+b$ from their denominator For correct integration	
			$=\frac{1}{4}\left(\tan^{-1}1-\tan^{-1}0\right)=\frac{1}{16}\pi$	M1	For substituting limits in any $\tan^{-1}$ expression	
				A1 **[5]**	For correct value	

Question			Answer	Marks	Guidance	
3			$\dfrac{2x^3+x+12}{(2x-1)\left(x^2+4\right)} \equiv A+\dfrac{B}{2x-1}+\dfrac{Cx+D}{x^2+4}$	B1	For correct form soi (*A* can be P*x* + Q, but not 0)	
			$2x^3+x+12 \equiv$ $A(2x-1)\left(x^2+4\right)+B\left(x^2+4\right)+(Cx+D)(2x-1)$	M1	For multiplying out from their form	
			$A = 1, B = 3$	B1	For either *A* or *B* correct (dep on 1st B1)	
			$x^3: \ 2=2A \quad x^2: \ 0=-A+B+2C$ $x^1: 1=8A-C+2D \qquad x^0: 12=-4A+4B-D$	M1	For equating at least 2 coefficients (or substitute two values for *x* or one of each)	
			$C = -1, D = -4$	A1A1	For *C*, *D* correct	
			$\Rightarrow 1+\dfrac{3}{2x-1}+\dfrac{-x-4}{x^2+4}$	A1 **[7]**	For correct expression **www** **SC4** $\Rightarrow \dfrac{3}{2x-1}+\dfrac{x^2-x}{x^2+4}$	
			ALT: Divide out as not proper $\Rightarrow 1+\dfrac{x^2-7x+16}{(2x-1)\left(x^2+4\right)}$	B1	Divide out	
			$=1+\dfrac{A}{2x-1}+\dfrac{Bx+C}{x^2+4}$	B1	Writing in this form including 1	
			$x^2-7x+16 \equiv A(x^2+4)+(Bx+C)(2x-1)$	M1	For multiplying out from their form	
			$x^2: 1=A+2B \qquad x: \ -7=-B+2C$ $1: 16=4A-C$	M1	For equating at least 2 coefficients (or substitute two values for *x* or one of each)	
			$\Rightarrow A=3, \ B=-1, \ C=-4$	A1 A1	*B* correct *C* correct	
			$\Rightarrow 1+\dfrac{3}{2x-1}+\dfrac{-x-4}{x^2+4}$	A1	For correct expression **www**	

Question			Answer	Marks	Guidance	
4	(i)		Given expression is sum of areas of rectangles of width $\frac{1}{n}$, heights $e^{-1/x}$ Given integral is area under the curve which is clearly greater	B1 B1 **[2]**	For identifying rectangle widths and heights For correct explanation of lower bound	
4	(ii)		Upper bound = $\frac{1}{n}\left(e^{-n}+e^{-\frac{n}{2}}+e^{-\frac{n}{3}}+\cdots+e^{-\frac{n}{n-1}}+e^{-1}\right)$	M1 A1 **[2]**	For using n upper rectangles soi by e^{-n} and e^{-1} For correct expression	
4	(iii)		Lower bound = 0.104(31) Upper bound = 0.196(28)	B1 B1 **[2]**	For correct value For correct value – accept 0.197	
4	(iv)		$\frac{1}{n}e^{-1} < 0.001$ $\Rightarrow n > \frac{1000}{e} = 367.879$ $\Rightarrow$ least $N = 368$	B1 M1 A1 **[3]**	For a correct statement (includes <) For rearranging (ignore < > = and allow RHS = $10^{\pm m}e^{\pm 1}$) For correct value	
5	(i)		$x_{n+1} = x_n - \frac{x_n^3 - k}{3x_n^2}$ $\Rightarrow x_{n+1} = \frac{2x_n^3 + k}{3x_n^2}$	M1 A1 **[2]**	For correct $\frac{f(x)}{f'(x)}$ seen (x or x_n) For simplification to **AG** (x_n and x_{n+1} required)	

Question			Answer	Marks	Guidance	
5	**(ii)**			B1 M1 A1 **[3]**	For correct curve with α (or $\sqrt[3]{k}$) and $-k$ marked For a suitable tangent shown with x_1 and x_2 marked such that $\lvert\alpha - x_2\rvert > \lvert\alpha - x_1\rvert$	Curve looks like cubic with one pt of inflection (g not nec. 0) at y axis
5	**(iii)**		$\alpha = \sqrt[3]{100}$ $x_2 = 4.66667$ $x_3 = 4.64172$	B1 B1 B1 **[3]**	For correct α (Condone x =...) For correct x_2 (to at least 5dp) For correct x_3 (to at least 5dp)	
5	**(iv)**		$e_1 = -0.35841$, $e_2 = -0.02508$, $e_3 = -0.00013$ $\frac{{e_2}^3}{e_1^2} = -0.00012$	M1 A1 A1 **[3]**	For calculating e_1, e_2, e_3 from α or something better than x_3 All correct to 5 dp For obtaining –0.00012 **SC2** for consistently without –ve signs	
6	**(i)**		$\cos y = x \Rightarrow -\sin y \frac{dy}{dx} = 1$ $\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$ – sign since $\frac{dy}{dx} < 0$ (e.g. by graph)	M1 A1 B1 **[3]**	For differentiating $\cos y$ wrt x For using $\cos^2 y + \sin^2 y = 1$ to obtain **AG** For justification of $+\sqrt{\ }$ taken **SC1** if in fractions $\frac{14}{3}$ and $\frac{2047}{441}$	

Question			Answer	Marks	Guidance	
6	(ii)		$\frac{dy}{dx} = -\frac{-2x}{\sqrt{1-(1-x^2)^2}}$	M1	For differentiating $\cos^{-1}(1-x^2)$ (as a function of a function)	
				A1	For correct $\frac{dy}{dx}$ (unsimplified)	
			$= \frac{2x}{\sqrt{2x^2-x^4}} = \frac{2}{\sqrt{2-x^2}}$	A1	For correct $\frac{dy}{dx}$ (simplified)	
			$\frac{d^2y}{dx^2} = 2.-\frac{1}{2}.-2x(2-x^2)^{-\frac{3}{2}} = \frac{2x}{(2-x^2)^{\frac{3}{2}}}$	M1	For differentiating $\frac{dy}{dx}$ using chain rule correctly (or product or quotient if y' is wrong)	
			$\Rightarrow (2-x^2)\frac{d^2y}{dx^2} = \frac{2x}{\sqrt{2-x^2}} = x\frac{dy}{dx}$	A1	For verification of **AG**	
				[5]		
7	(i)		$x = \sinh y = \frac{e^y - e^{-y}}{2}$	M1	For correct expression for sinh y and attempt to obtain quadratic	
			$\Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2+1}$	A1	For correct solution(s) for e^y	
			reject – sign as $e^y > 0 \Rightarrow y = \ln\left(x+\sqrt{x^2+1}\right)$	A1	For justification of + sign to **AG**	
				[3]		
			Alt: $\sinh y + \cosh y = e^y$ $\sinh y = x \Rightarrow \cosh y = \pm\sqrt{x^2+1}$ reject -ve sign as $e^y > 0$ $\Rightarrow e^y = x+\sqrt{x^2+1} \Rightarrow y = \ln\left(x+\sqrt{x^2+1}\right)$			

Question			Answer	Marks	Guidance	
7	(ii)		$\ln\left(x+\sqrt{x^2+1}\right)-\ln\left(x+\sqrt{x^2-1}\right)=\ln 2$	M1	For stating both ln expressions and attempting to exponentiate	Removing lns is not an attempt to exponentiate
			$\Rightarrow \dfrac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}}=2$			
			$\Rightarrow \sqrt{x^2+1}-2\sqrt{x^2-1}=x$	A1	For correct equation **AG**	
				M1	For attempting to square once	
			$\Rightarrow 4x^2-3=4\sqrt{x^4-1}$	A1	For a correct equation with $\sqrt{}$ as subject	
			$\Rightarrow 24x^2=25\Rightarrow x=\dfrac{5}{\sqrt{24}}\left(=\dfrac{5}{12}\sqrt{6}\right)$	A1	For correct x and no others **isw**	
				[5]		
8	(i)		$2\cos^2\alpha=2\sin 2\alpha=4\sin\alpha\cos\alpha$	M1	For equation in $\cos\alpha$ and $\sin\alpha$ (only - ie dealing with $\sin 2\alpha$	
			$\Rightarrow \tan\alpha=\frac{1}{2}$	A1	leading to **AG**(θ may be used instead of α) **SR** Allow verification only if exact	
				[2]		
8	(ii)		$\text{Area}=\frac{1}{2}\int_0^{\alpha} r_2^2\,d\theta+\frac{1}{2}\int_{\alpha}^{\frac{1}{2}\pi} r_1^2\,d\theta$	M1	For both integrals added with limits soi Allow θ for α , and reversal of r^2 terms	
			$=\frac{1}{2}\int_0^{\alpha} 2\sin 2\theta\,d\theta+\frac{1}{2}\int_{\alpha}^{\frac{1}{2}\pi} 1+\cos 2\theta\,d\theta$	M1	For using $2\cos^2\theta=1+\cos 2\theta$ in 2nd integral	
			$=\left[-\frac{1}{2}\cos 2\theta\right]_0^{\alpha}+\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_{\alpha}^{\frac{1}{2}\pi}$	M1	For $k\cos 2\theta$ as first integrated term	
			$=\left(-\frac{1}{2}\cos 2\alpha+\frac{1}{2}\right)+\left(\frac{1}{4}\pi-\frac{1}{2}\alpha-\frac{1}{4}\sin 2\alpha\right)$	A1	For correct first area	
			$=\left(-\frac{1}{2}\left(1-2\sin^2\alpha\right)+\frac{1}{2}\right)+\left(\frac{1}{4}\pi-\frac{1}{2}\alpha-\frac{1}{2}\sin\alpha\cos\alpha\right)$	A1	For correct second area	
			$=\frac{1}{5}+\frac{1}{4}\pi-\frac{1}{2}\alpha-\frac{1}{2}.\frac{1}{\sqrt{5}}.\frac{2}{\sqrt{5}}$	M1	For using Pythagoras to find $\sin\alpha$ or $\cos\alpha$ ***OR*** t formula for $\cos 2\alpha$ or $\sin 2\alpha$	
			$=\frac{1}{4}\pi-\frac{1}{2}\alpha$	A1	For simplification to **AG**	
				[7]		

Question			Answer	Marks	Guidance	
9	(i)		$\tanh(\ln n) = \dfrac{e^{\ln n} - e^{-\ln n}}{e^{\ln n} + e^{-\ln n}}$	M1	For definition of $\tanh(\ln n)$ seen Or working with $\tanh(\ln n) = x$, definition of $\tanh^{-1} x$ seen	
			$= \dfrac{n - \frac{1}{n}}{n + \frac{1}{n}} = \dfrac{n^2 - 1}{n^2 + 1}$	A1	For simplification to **AG** **SC1** $\tanh(\ln n) = \dfrac{e^{\ln n} - e^{-\ln n}}{e^{\ln n} + e^{-\ln n}} = \dfrac{e^{2\ln n} - 1}{e^{2\ln n} + 1} = \dfrac{n^2 - 1}{n^2 + 1}$	
				[2]		
9	(ii)		$I_n - I_{n-2} = \int_0^{\ln 2} \left(\tanh^n u - \tanh^{n-2} u\right) \mathrm{d}u$ $= \int_0^{\ln 2} \tanh^{n-2} u \left(\tanh^2 u - 1\right) \mathrm{d}u = -\int_0^{\ln 2} \tanh^{n-2} u \operatorname{sech}^2 u \, \mathrm{d}u$	M1	For factorising and replacing $(\tanh^2 u - 1)$ by $\pm \operatorname{sech}^2 u$ (or similarly considering I_n)	
			$\Rightarrow I_n - I_{n-2} = -\left[\frac{1}{n-1} \tanh^{n-1} u\right]_0^{\ln 2}$	A1	For correct integrated term	
			$\Rightarrow I_n - I_{n-2} = -\frac{1}{n-1}\left(\frac{3}{5}\right)^{n-1}$	A1	For simplification to **AG**	
				[3]		
9	(iii)		$I_1 = \int_0^{\ln 2} \tanh u \, \mathrm{d}u = \left[\ln \cosh u\right]_0^{\ln 2}$	M1	For integration to $k \ln \frac{\cosh u}{\sinh u}$	
			$= \ln\left(\cosh(\ln 2)\right) = \ln \dfrac{e^{\ln 2} + e^{-\ln 2}}{2} = \ln \frac{5}{4}$	M1	For simplifying $\frac{\cosh}{\sinh}(\ln 2)$	
				A1	For correct value of I_1	
			$I_3 = I_1 - \frac{1}{2}\left(\frac{3}{5}\right)^2 = -\frac{9}{50} + \ln \frac{5}{4}$	B1ft	For correct I_3. FT from I_1 **SC** $I_3 = -\frac{9}{50} + \ln(\cosh(\ln 2))$ **M1 B1ft**	
				[4]		
9	(iv)		$\left(I_n - I_{n-2}\right) + \left(I_{n-2} - I_{n-4}\right) + \ldots + \left(I_3 - I_1\right)$ $= I_n - I_1 = -\left(\dfrac{1}{n-1}\left(\dfrac{3}{5}\right)^{n-1} + \dfrac{1}{n-3}\left(\dfrac{3}{5}\right)^{n-3} + \ldots + \dfrac{1}{2}\left(\dfrac{3}{5}\right)^2\right)$	M1	For attempting to sum equations of the form of **(ii)** and cancelling soi	
			$\Rightarrow \frac{1}{2}\left(\frac{3}{5}\right)^2 + \frac{1}{4}\left(\frac{3}{5}\right)^4 + \frac{1}{6}\left(\frac{3}{5}\right)^6 + \ldots = I_1 = \ln \frac{5}{4}$	A1ft	For correct answer ft from I_1	
				[2]		

Alternative to Q9(ii)

Question			Answer	Marks	Guidance	
9	(ii)		$I_n = \int_0^{\ln 2} \tanh^n u \, du = \int_0^{\ln 2} \tanh^{n-2} u . \tanh^2 u \, du$	M1	For attempt to integrate by parts.	
			$= \int_0^{\ln 2} \tanh^{n-2} u . \left(1 - \text{sech}^2 u\right) \, du$			
			$= \int_0^{\ln 2} \tanh^{n-2} u . \, du - \int_0^{\ln 2} \tanh^{n-2} u \, \text{sech}^2 u \, du$			
			$\Rightarrow I_n = I_{n-2} - \left[\dfrac{\tanh^{n-1} u}{n-1}\right]_0^{\ln 2}$	A1	For correct integrated term	
			$\Rightarrow I_n - I_{n-2} = -\dfrac{\tanh^{n-1}(\ln 2)}{n-1}$			
			$= -\dfrac{1}{n-1}\left(\dfrac{2^2-1}{2^2+1}\right)^{n-1} = -\dfrac{1}{n-1}\left(\dfrac{3}{5}\right)^{n-1}$	A1	For simplification to **AG**	
				[3]		